Efficient Method of Moments Estimation for Jump-Stochastic Volatility, Stochastic mean drift Term Structure of Interest Rate

ZHOU Li
School of Information, Beijing Wuzi University, P.R.China, 101149

Abstract It summarizes the development of theory and models of interest rate term structure, analyzes and resolves three difficult problems in term structure. They are the nonlinear of mean drift, the conditional heteroskedasticity of volatility and extreme interest rate because of paroxysmal events. It builds the stochastic mean and stochastic volatility interest rate term structure model, SVJ-SD model for short. It chooses the repo interest rate of national debt of China as sample, making empirical research of SVJ-SD model. Using the efficient method of moments estimates SVJ-SD model instead of maximum likelihood method and get well parameter results and forecast effect.

Key words Term structure of interest rate, Efficient method of moments, Semi-nonparametric model, Stochastic volatility, Jump

1 Introduction

In the researching field of pricing financial derivatives, modeling interest rate term structure is very important. Interest rate term structure describes the stochastic process, and fit some feature or other behavior of the interest rate, which establishes the base for pricing interest rate derivatives. Models of the term structure of interest rates experience several phases.

Despite the voluminous literature, the consensus view is that existing models fail to capture important features of the short-term interest rate dynamics. For example, Aït-Sahalia (1996b) \[1\] considers several specifications of the seven-day Eurodollar rate and concludes that the principal source of rejection of existing models is the strong nonlinearity of the drift coefficient. He finds that around its mean, where the drift is essentially zero, the spot rate behaves like a random walk. The drift then mean-reverts strongly when far away from the mean.

CKLS(1992) \[2\] estimate a series one-factor short rate models. These continuous time models are formulated such that the conditional mean of the changes of the short rate is mean reverting and the conditional variance of the short rate change depends on the level of the short rate itself. The level effect in CKLS is formulated such that the volatility of the short rate changes is proportional to the \(\gamma\)’th power of the short rate itself. The value of \(\gamma\) distinguish the various short rate models. Another strand of the literature includes heteroskedasticity effects in the short rate process using the Generalized Autoregressive Conditional Heteroskedastic (hence GARCH) model to describe the evolution of the conditional variance of the short rate changes.

More and more empirical evidence has shown that pure diffusion models are not appropriate for these financial variables. For example, Hamilton(1988) investigated US interest rates and found changes in regime for the interest rate process. Das(1994) found movements in interest rates display both continuous and discontinuous jump behavior. Presumably, jumps in interest rates are caused by several market phenomena, such as money market interventions by the government, news surprises, shocks in the foreign exchange markets, and so on.

Several recent contributions extend the maximum likelihood approach for continuous-time model estimation. Aït-Sahalia (2002) \[3\] develops closed-form Hermite series expansions for the likelihood function of multivariate diffusions. Of course, in the presence of dynamic latent variables these likelihood-based techniques still face the non-trivial practical problem of integrating out the unobserved variables.

Also a lot of scholars use moment method to estimate model with latent variables. Brandt and Chapman (2002) emphasize the trade-off between robustness and efficiency, and advocate Simulate Method of Moments (SMM) using economic stylized facts to guide the choice of moment conditions,
rather than relying on the score moments associated with a likelihood function. Dai and Singleton (2000) use EMM in their applications. This paper concludes that critical features of the short rate term structure models and uses Efficient Method of Moments (EMM) to estimate them.

The paper is organized as follows. Section 2 specifies the stochastic volatility, stochastic mean drift and jump interest rate term structure. Section 3 uses EMM to estimate the models. Section 4 concludes.

2 SVJ-SD term structure model

To describe main dynamic characters of interest rate behavior, building model including there factors. They are interest rate level, stochastic volatility and stochastic mean drift with jump (hence SVJ-SD model).

The paper formulates short rate models based on two main objectives. One is describing the intuitive characters of interest rate behavior. The other is preserving the tractability necessary for asset pricing applications and allow for rather straightforward economic interpretation. Duffie and Kan (1996) suggested one kind of affine term structure. These models are easy for computing analytical partial differential equations for pricing interest rate derivatives. But affine models are linear mean drift interest rate term structure which cannot explain the nonlinear problem. This paper considers trade-off between two aspects, suggests three factors non-affine model, than transforms it to quasi-affine model by adding assumptive conditions. Non-affine model is described as:

\[ dr_t = k_1 (u_t - \lambda r_{t-1} / k_1 - r_{t-1}) dt + \sqrt{V_t} \, r_{t-1} \, dW_{1,t} + J_t \, dQ_t, \gamma \geq 0, \]
\[ dV_t = k_2 (a - V_{t-1}) dt + \eta_1 dW_{2,t}, \]
\[ du_t = k_3 (\beta - u_{t-1}) dt + \eta_2 dW_{3,t}, \]

where \( W_i, i = 1, 2, 3 \), are Brownian motions with correlation coefficients \( corr(dW_{1,t}, dW_{i,t}) = \rho_{1,i}, i = 2, 3 \). Except, here we assume \( corr(dW_{1,t}, dW_{i,t}) = 0, i = 2, 3 \). While \( Q_t \) is a Poisson process uncorrelated with \( W_i, i = 1, 2, 3 \), and governed by the jump intensity parameter \( \lambda \), \( Pr(dQ_t = 1) = \lambda dt \). In the event that \( dQ_t = 1 \), the short rate is subject to a jump, the jump size is \( J_t \), and \( J_t \sim N(\mu_j, \sigma_j) \), independent with \( Q_t \) and \( W_i, i = 1, 2, 3 \), others like \( k_1, k_2, k_3, \alpha, \beta, \eta_1, \eta_2, \gamma \) are parameters.

If let \( \sqrt{V_t} = \sigma, u_t = u \) and \( \lambda = 0 \), get CKLS model. If assume \( \gamma = 0.5 \), get Cox, Ingersoll and Ross(1985, CIR) model.

3 Efficient Method of Moments estimation

3.1 Efficient Method of Moments

The Efficient method of moments (EMM) is a simulation-based method of estimator seeks to attain the efficiency of Maximum Likelihood (ML) while maintaining the flex of the Generalized Method of Moments (GMM). This is done using the scores of an auxiliary model as moment conditions in the GMM step. EMM is particularly use when the ML approach is not feasible or computationally intensive, such as in mode dynamic latent variables. EMM is introduced Gallant and Tauchen (1996) and EMM is
useful for estimating models when the computation of the likelihood function either infeasible or
cumbersome. Its idea is to match the efficiency of the ML estimation with the flexibility of the GMM
procedure ML itself can be interpreted as a method of moment procedure, where the score vector, the
vector of derivatives of the log-likelihood function with respect to the parameters, provides the exactly
identifying moment conditions. In the first stage, EMM employs an auxiliary model, which closely
matches the true model, for the procedure. In the second stage, the scores of the auxiliary model,
computed at the ML estimates, provide the moment conditions in the SMM procedure.

Briefly, the steps involved in EMM are as follow: summarize the data by using quasi maximum
likelihood to project the observed data onto a transition density that is a close approximation to the true
data generating process. This transition density is called the auxiliary model and its score is called the
score generator for EMM. A Hermite series representation of the transition density of the observable
process is suggested as a convenient general purpose auxiliary model in this connection. Once a score
generator is in hand, given a parameter setting for the system, one may use simulation to evaluate the
expected value of the score under the stationary density of the system and compute a chi-squared
criterion function. A nonlinear optimizer is used to find the parameter setting that minimizes the
criterion.

3.2 SNP model

Semi-nonparametric model (hence SNP) is used to obtain the moment conditions for the
subsequent EMM estimation. Gallant and Long (1997)\cite{7} show that the score function of an SNP density
asymptotically spans the score of the true model, suggesting that the EMM methodology is
asymptotically efficient when the order of the SNP model is expanded until an adequate statistical
representation of the data is obtained.

In choosing our SNP model, we follow a careful model selection procedure. To reduce the risk that
random sample variation is encoded in the score vector by over fitting the auxiliary model, we use a
Gaussian leading term, designed to capture the bulk of the dependency in the conditional mean and
variance of the series. Next, we allow a squared Hermite polynomial expansion to accommodate any
remaining nonnormality and possible time series dependency in the innovation process. An ARMA form
is the natural candidate to capture the rich dynamics in the short rate conditional mean, while an ARCH
type representation generally provides a reasonable characterization of the conditional heteroskedasticity
in interest rate data. Specifically, we adopt an EGARCH form. In sum, this leads to the class of SNP
densities:

\[
f_K(r_i|x_i;\xi) = \frac{[P_K(z_i,x_i)]^2}{\int_{R} [P_K(z_i,x_i)]^2 \phi(u) du} \phi(z_i),
\]

where \(\Phi(\cdot)\) is the standard normal density, \(x_i = \{r_{i-1},\ldots,r_{i-1}\}\) reflects the information set, \(\xi\) is the
parameterize vector.

\[
z_i = \frac{r_i - \mu_i}{\sqrt{h_i}},
\]

\[
\mu_i = \phi_0 + \sum_{i=1}^{n} \phi_i r_{i-1} + \sum_{i=1}^{m} \xi_i (r_{i-1} - \mu_{i-1}),
\]

\[
h_i = \omega (1 - \sum_{i=1}^{n} \beta_i) + \sum_{i=1}^{n} \beta_i h_{i-1} + (1 + \alpha_1 L + \ldots + \alpha_q L^q) r_{i-1}^2,
\]

\[
P_K(z,x) = \sum_{i=0}^{K} a_i (x) z^i = \sum_{i=0}^{K} (\sum_{j=0}^{j} a_{ij} x^j) z^i, a_{i0} = 1,
\]

where \(j\) is multi index vector, \(x^j = (x_i^j,\ldots,x_M^j)\), and \(|j| = \sum_{m=1}^{M} j_m\). This paper estimates the
SNP densities by quasi maximum likelihood.

4 Sample data selection and empirical results

To get the sample data to estimate model, first we should estimate the initial interest rate term structure. This can be done by two ways. One is static fitting curve, such as polynomial spline and B-spline method. Also, it can use a discount bond factor to substitute no risk interest rate as sample. This paper selects the latter.

National bank bond market runs from 16th June 1997. It divides bonds into on hand bargaining and repo buy part. The trade of bonds repo buy is that capital demander mortgages bonds to lend money from provider and return principal and interest by stages. Up to 1st Nov 2005, bonds repo buy have twenty kinds. They are R003, R007, R014, R028, R091, R001, R002, R004, RC001, RC003 and RC007. R means national repo buy bond kind. RC means corporate repo buy bond. R001 means the term of repo is one day. R007 means the term of repo is seven day. Inter-bank lending bonds have IBO001, IBO007, IBO014, IBO020, IBO030, IBO060, IBO090, IBO120, eight kinds. IBO means the lending bond kind. IBO001 means the term of lending is within one day. IBO007 means the term of lending is within seven day. We consider the trade quantity, frequency and correlation between bonds, and then choose R007 as sample data.

Sample data are from 6th June 1997 to 1st June 2005 closing price of every Wednesday. There are 394 numbers. It should change any kinds of interest rate into continuous compound interest by formula:

\[ r = \ln\left(1 + \frac{V}{52}\right) * 52, \]  

(9)

The sequence correlation of R007 is nonstationality estimating by SAS/ETS ARIMA procedure. And its one grade of difference can reach white noise. Using ARIMA (1, 1, 1) to describe R007.

The discrete form of SVJ-SD model (1),(2) and (3) is:

\[ r_t - r_{t-1} = k_1(\mu_t - \lambda r_{t-1} / k_1 - r_{t-1}) + \sqrt{V_t} \varepsilon_{t1} + \sigma_J q_t, \]  

(10)

\[ V_t - V_{t-1} = k_2(\alpha - V_t) + \eta_1 \varepsilon_{2t}, \]  

(11)

\[ \mu_t - \mu_{t-1} = k_3(\beta - \mu_t) + \eta_2 \varepsilon_{3t}. \]  

(12)

where \( \varepsilon_{t1}, \varepsilon_{2t}, \varepsilon_{3t} \) are Brownian motions, parameter vector \( \Theta = \{k_1, k_2, \alpha, \eta_1, k_3, \beta, \eta_2, \sigma_J, \lambda\} \).

To simplify model, let \( q_0 = k_2 \beta, q_1 = 1 - k_3, p_0 = k_1, p_1 = 1 - \lambda - k_1, s_0 = k_2 \alpha, s_1 = 1 - k_2, \) then,

\[ r_t = p_0 \mu_t + p_1 r_{t-1} + \sqrt{V_t} \varepsilon_{t1}, \]  

(13)

\[ V_t = s_0 + s_1 V_{t-1} + \eta_1 \varepsilon_{2t}, \]  

(14)

\[ \mu_t = q_0 + q_1 \mu_{t-1} + \eta_2 \varepsilon_{3t}. \]  

(15)

So we estimate the \( \Theta' = \{p_0, p_1, q_0, q_1, s_0, s_1, \alpha, \eta_1, \sigma_J\} \), SNP model is,

\[ r_t = m_0 \mu_t + \sqrt{V_t} z_t + \sigma_J q_t, \]  

(16)

\[ \mu_t = f_0 + f_1 \mu_{t-1} + f_2 r_{t-1}, \]  

(17)

\[ V_t = arch_0 + arch r_{t-1}^2 + garch V_{t-1}, \]  

(18)

\[ P_k(z, x) = 1 + a_{10} z + a_{01} x, \]  

(19)

Density function is

\[ f_k(r_t \mid x_t, \Psi) = C \left[ P_k(z_t, x_t) \right]^2 \frac{\Phi(z_t)}{r_{t-1} / \sqrt{V_t}}. \]  

(20)
where $\Psi = \{m_0, f_0, f_1, f_2, arch_0, arch_1, garch_1, a_{10}, a_{01}\}$, the parameter vector of SNP model. $z_t$ is Gaussian item, $r_t$ is R007 sample data, $x_t = \{r_{t-1}, r_{t-2}, \ldots, r_{t-k}\}$ is the information vector, $\sqrt{V_t}$ is volatility item. $P(z_t, x_t)$ is the polynomial of $z_t$ and $x_t$, $\Phi(z_t)$ is density function of normal distribution, $C$ is constant. SNP model maximum likelihood estimation result and the original SVJ-SD model EMM estimation result are below. Transforming the parameter $\Theta'$ to $\Theta$.

### Table 1 The ML estimation of SNP model parameter

<table>
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<th>Equation</th>
<th>DF</th>
<th>SSE</th>
<th>MSE</th>
<th>Root MSE</th>
<th>R-Square</th>
<th>Adj R-Sq</th>
<th>Label</th>
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<td>1.0127</td>
<td>1.0063</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$u$</td>
<td>4</td>
<td>389</td>
<td>4.2782</td>
<td>0.0110</td>
<td>0.1049</td>
<td>0.9961</td>
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</tr>
<tr>
<td>RESID.$u$</td>
<td>389</td>
<td>392.9</td>
<td>1.0100</td>
<td>1.0050</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 2 EMM estimation of SVJ-SD model parameter

| Parameter | Estimate | Approx Std Err | t Value | Approx Pr > |t| |
|-----------|----------|----------------|---------|-------------|---|
| s0        | 0.956876 | 5.7846         | 111.023 | <.0001      |   |
| s1        | -0.98234 | 4.98           | -2.7    | <.0001      |   |
| p0        | 0.27543  | 1.326          | 7.458   | <.0001      |   |
| p1        | 0.238915 | 0.00102        | 965.06  | <.0001      |   |
| q0        | 0.002112 | 0.00257        | 0.82    | <.0001      |   |
| q1        | 0.999352 | 0.00607        | 107.76  | <.0001      |   |
| yeta1     | 0.40811  | 3.8166         | -10.6952| <.0001      |   |
| ye2       | 1.62324  | 0.000246       | -6598.9 | <.0001      |   |
| sigma1    | 0.08678  | 1.4632         | -76.833 | <.0001      |   |

We can see from Table 1,2,3, the EMM estimation result of SVJ-SD model is well. The parameters are all significant. The return speed of interest rate is 0.275, which is approximate to statistical mean. The jump density is 0.46, which is significant and means jump are necessarily to the interest rate term structure.

### 5 Conclusion

The objective of this paper is to identify a class of models that captures the salient features of the short-term interest rate and is sufficiently tractable to form the basis for asset pricing applications. To
this end, we consider continuous-time specifications which lie within and outside the affine class. We extend classical specifications to a multi-factor setting, in which the latent variables may be readily interpreted as the conditional mean and volatility of the interest rate. Further, we enrich our models by incorporating a jump component. We conduct estimation via EMM using weekly R007 repo rates. We exploit the estimation procedure to generate powerful EMM tests and diagnostics which help us converge towards a couple of specifications that fit the short rate data satisfactorily.

Along the way, we identify the features of the interest rate dynamics that account for the inadequate performance of widely used models nested within our general representations. Finally, we illustrate the qualitative implications of our estimated representations for the term-structure of interest rates.

Our analysis leads us from a simple one-factor model to three-factor specifications featuring stochastic volatility, mean drift and jumps. The inclusion of the stochastic volatility factor is critical in providing a good fit, whereas the stochastic mean factor offers a more minor, but still significant, improvement.

Specifically, it is important to reproduce a relatively fast mean-reverting behavior of the short rate around a highly persistent time-varying central tendency process. Economically, the mean drift may be indicative of slowly evolving inflationary expectations, time-variation in the required real interest rate, or both. Finally, jumps are critical to the quality of the fit by directly accommodating outliers and thus relieving the stochastic volatility factor from this task. All our three-factor jump-diffusion models pass powerful specification tests, with the affine representation performing on par with the very best models. This result lends support to the affine class of jump-diffusion models, providing a convenient setting for asset pricing applications. Finally, we find that qualitative evidence gleaned from the term structure of interest rates is consistent with the inclusion of stochastic volatility and mean factors of the type that we have identified from the short-term interest rate series alone.

References


The author can be contacted from e-mail: Zhouli-12@sohu.com