On Nash Equilibrium of the Abstract Economy or Generalized Game in H-Spaces

Xie Fang, Xu Yuguang
Department of Mathematics, Kunming Teacher’s College, Kunshi Road No. 2, Kunming 650031, P. R. China

Abstract The purpose of this paper is to introduce the H-space and to establish a theorem for the Nash equilibrium of a generalized game (or an abstract economy). The result presented in this paper, modify and extend the corresponding results given in the literatures, so that this new theorem are applied to the equilibrium problems of the game (or the economy) in H-space.

Key words H-space; H-convex subset; H-quasi-convex function; H-quasi-concave function; Nash equilibrium

1. Introduction and Preliminaries
A game is a situation in which several players each have partial control over some outcome and generally have conflicting preferences over the outcome. The set of choices under player \( i \)'s control is denoted \( X_i \). Elements of \( X_i \) are called strategies and \( X_i \) is \( i \)'s strategy set. Letting \( I \) denote the set of players, \( X = \prod_{i \in I} X_i \) is the strategy vectors. We will describe a generalized qualitative game or abstract economy by \( B = \{X_i, A_i, f_i : i \in I\} \) Where \( I \) is finite or infinite ( countable or uncountable ) set of players and agents, and for each \( i \in I \), \( X_i \) is the non-empty set of actions, the strategy set or choice set; \( A_i : X = \prod_{i \in I} X_i \rightarrow 2^{X_i} \) is the preference correspondence (set valued mapping) and \( f_i : X \rightarrow R \) is the utility or pay off function. In generally, \( X_i \) will be a subset of a topological vector space for each \( i \in I \).

A point \( \bar{x} = (\bar{x}_i, \bar{x}') \in X_i \times \prod_{i \neq j} X_j \) is called an equilibrium point of \( B \) if

\[
f_i(\bar{x}) = \sup_{u \in A_i(\bar{x})} f_i(u, \bar{x}') \quad \forall i \in I.
\]

The definitions of an abstract economy and an equilibrium coincide with the standard ones and for further information of this topic, the reader is referred to Shafer-Sonnenschein[1].

In 1950, J. Nash proves the existence of equilibrium for games where the player’s preferences are representable by continuous quasi-concave utilities and the strategy sets are simplexes. Debreu, in 1952, he proves the existence of equilibrium for the generalized qualitative game or abstract economy. He assumes that strategy sets are contractible polyhedral and that the feasibility correspondences have closed graph, and the maximized utility is continuous and that the set of utility maximizes over each constraint set is contractible.

In 1975, an important theorem is proved by Shafer and Sonnenschein as follows.

**Theorem S-S.** Let \( B = \{X_i, A_i, f_i : i \in I\} \) be an abstract economy. For each \( i \in I \) the conditions as follows are satisfied

(i) \( X_i \subset R \) is nonempty, compact and convex;

(ii) \( A_i : X \rightarrow 2^{X_i} \) is a continuous correspondence with nonempty compact convex values;

(iii) \( Gr f_i \) is open in \( X \times X_i \);

(iv) \( x_i \notin co f_i(x) \) for all \( x \in X \)

then there is an equilibrium point of \( B \).
The purpose of this paper is to introduce the H-space and to establish an existence theorem for the Nash equilibrium of a generalized game (or an abstract economy). The result, presented in this paper, modify and extend the corresponding results given in the literatures so that it are applied to more general equilibrium problems of the game (or the economy) in H-space.

To set the framework, we recall the notation of H-space introduced by C. Horvath in 1987 as follows.

**Definition 1.1.** Let $X$ be a topological space and $F(X)$ a family of non-empty finite subsets of $X$. Let $\{\Gamma_d\}$ be a family of non-empty contractible subsets of $X$ indexed by $A \in F(X)$ such that $\Gamma_d \subset \Gamma_{d'}$ whenever $A \subset A'$. The pair $(X, \{\Gamma_d\})$ is called an H-space.

**Definition 1.2.** Let $\left\{ (X_i, \{\Gamma_d\}_i) : i \in I \right\}$ be a family of H-spaces where $I$ is a finite or infinite index set. It is Obvious that the Cartesian product $X := \prod_{\Gamma_d} \{ (X_i, \{\Gamma_d\}_i) : i \in I \}$ is still an H-space (see Lemma 1.1 in [3]).

**Definition 1.3.** The set $C \subset X$ is called H-convex in $(X, \{\Gamma_d\})$ if $C \subset \Gamma_d$ for every non-empty finite subset $C \subset X$.

**Definition 1.4.** Let $\left\{ (X_i, \{\Gamma_d\}_i) : i \in I \right\}$ be an H-space and $Y$ a topological space. Let $f : X \times Y \to R \cup \{\pm \infty\}$ be a functional and $\lambda \in R$. For each fixed $y \in Y$, $f(x, y)$ is called H-quasi-convex (or H-quasi-concave) in $X$ if and only if the set $\{x \in X : f(x, y) < \lambda\}$ (or $\{x \in X : f(x, y) > \lambda\}$) is H-convex for any $\lambda \in R$.

Throughout this paper, all topological space are considered as Hausdorff spaces.

2. Nash equilibrium of the abstract economy or generalized game in H-space

In order to prove our main theorem, we first state a lemma which had been proved in [3].

**Lemma 2.1.** Let $\left\{ (X_\alpha, \{\Gamma_d\}_\alpha) : \alpha \in I \right\}$ be a family of compact H-spaces, where $A \in F(X)$ and $I$ is an index set containing at least two elements. Denote $X = \prod_{\Gamma_d} \{ X_\alpha : \alpha \in I \}$ and $X^\alpha = \prod_{\Gamma_d} \{ X_\beta : \beta \in I, \beta \neq \alpha \}$.

For each $\alpha \in I$, $t_\alpha \in R$, let $f_\alpha : X \to R$ be a functional such that

2.1.1 for each $x = (x_\alpha, x^\alpha) \in (X_\alpha, X^\alpha)$, there is a $y_\alpha \in X_\alpha$ such that $f_\alpha(y_\alpha, x^\alpha) > t_\alpha$;

2.1.2 $f_\alpha(\bullet, y^\alpha)$ is H-quasi-concave for each fixed $y^\alpha \in X^\alpha$;

2.1.3 $f_\alpha(x_\alpha, \bullet)$ is lower semi-continuous for each fixed $x_\alpha \in X_\alpha$.

Then there exists a $\bar{x} = (\bar{x}_\alpha, \bar{x}^\alpha) \in X$ such that $f_\alpha(\bar{x}) > t_\alpha$ for each $\alpha \in I$.

**Lemma 2.2.** Let $X$ be a topological space and $I$ an index set containing at least two elements. Suppose that $\{ (X_i, \{\Gamma_d\}_i) : i \in I \}$ is a family of compact H-spaces and $\left\{ f_i : X_\alpha \to R : i \in I \right\}$ is a family of continuous functional. If $f_i(x_i, x^i)$ is H-quasi-concave in $X_i$ for each $i \in I$, then there exists a point $\bar{x} \in X$ such that
\[ f_i(\bar{x}) = \max_{u_i \in X_i} f_i(u_i, \bar{x}^i) \]

for all \( i \in I \).

**Proof.** Setting
\[
F_i(x) = f_i(x) - \max_{u_i \in X_i} f_i(u_i, x^i)
\]

for all \( i \in I \) and all \( x = (x_i, x^i) \in X \), then, for each \( i \in I \) by the uniform continuity of \( f_i \) on \( X \), we can easily see that \( \max_{u_i \in X_i} f_i(u_i, x^i) \) is a continuous function on \( X \). Consequently, \( F_i(x) \) is continuous on \( X \). Now, for each \( i \in I \) and for any fixed \( y^i \in X^i \), we have \( f_i(x_i, y^i) \) is H-quasi-concave in \( x_i \). Hence, \( F_i(x_i, y^i) \) is H-quasi-concave as well. In view of the compactness of \( X_i \), there exist an \( \bar{u}_i \in X_i \) such that \( \max_{u_i \in X_i} f_i(u_i, y^i) = f_i(\bar{u}_i, y^i) \). Consequently, we obtain that \( F_i(\bar{u}_i, y^i) > -\varepsilon \) for each \( \varepsilon > 0 \). By virtue of Lemma 2.1, there exists an \( \bar{x} \in X \) such that \( F_i(\bar{x}) > -\varepsilon \) \( \forall i \in I \).

Putting
\[ E_i(\varepsilon) = \{ x \in X : F_i(x) \geq -\varepsilon \} \]

for all \( i \in I \), and all \( \varepsilon > 0 \), from the continuity of \( F_i \), we know that \( E_i(\varepsilon) \) is a non-empty compact subset of \( X \). If \( \varepsilon_1 < \varepsilon_2 \), then it is clear that \( E(\varepsilon_1) \subset E(\varepsilon_2) \). Therefore, we have \[ \bigcap_{\varepsilon \in \mathbb{R}^+} E(\varepsilon) \neq \emptyset \]. It follows that there exists \( \bar{x} = (\bar{x}_i, \bar{x}^i) \in X \) such that \( F_i(\bar{x}) \geq 0 \) for each \( i \in I \), i.e.,
\[ f_i(\bar{x}) = \max_{u_i \in X_i} f_i(u_i, \bar{x}^i) \quad \forall i \in I \].

This completes the proof. \( \square \)

**Theorem 2.3.** Let \( B = \{ X_i, A_i, f_i : i \in I \} \) be an abstract economy or generalized qualitative game and \( I \) an index set containing at least two elements. For each \( i \in I \), if the following conditions hold:

(2.3.1) \( (X_i, \{ \Gamma_{ij} \}) \) is a compact H-space;
(2.3.2) \( A_i(x) = X_i, \quad \forall x \in X \);
(2.3.3) \( f_i : X \rightarrow R \) is continuous on \( X \) and H-quasi-concave on \( X_i \);

then there exists an equilibrium point of \( B \).

**Proof.** The conclusion of Theorem 2.3 comes from Lemma 2.2. In fact, \( X = \prod_{i \in I} X_i \) also is a compact H-space, \( \{ A_i : i \in I \} \) is a family of continuous correspondence with nonempty compact convex values and \( f_i(x_i, x^i) \) is H-quasi-concave in \( x_i \) for each \( i \in I \). It follows Lemma 2.2 that there exists a point \( \bar{x} = (\bar{x}_i, \bar{x}^i) \in X \) such that
\[ f_i(\bar{x}) = \max_{u_i \in X_i} f_i(u_i, \bar{x}^i) \quad \forall i \in I \].

I.e., \( \bar{x} \) is an equilibrium point of \( B = \{ X_i, A_i, f_i : i \in I \} \).

This completes the proof. \( \square \)

**Remark.** Theorem 2.3 is an existence theorem of the equilibrium with infinite agents on the strategy.
spaces without linear structure. Hence, it extend the corresponding results of J.P.Aubin [4] and Ky Fan [5].

References