Stock Futures Pricing Based on the Minimal Martingale Measure
In Discrete-time Incomplete Markets

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Abstract  The increasing interest in financial innovation of enterprises has heightened the need for the knowledge of accurate pricing for derivatives in actual discrete-time incomplete market, especially for futures, the most actively traded derivatives in China. Nevertheless, even contingent claim pricing in such markets have few previous researches concentrated on, quite apart from futures. This paper develops stock futures pricing in discrete-time incomplete markets based on the minimal martingale measure and the locally risk-minimizing strategy. The author establishes a specified market model, in which the minimal martingale measure is worked out. Under the measure the arbitrage-free pricing model for stock futures is obtained. By using data of several representative stock futures and stock index futures selected from American financial markets, an empirical test is implemented to investigate the efficiency of the model. The results indicate that the predicting prices given by the new model could fit the actual prices well, and compared to a traditional single-factor model, the new model is strongly possible to perform better in prediction. These results offer attractive possibilities for applications.

Key words  Discrete-time incomplete market, Minimal martingale measure, Locally risk-minimizing strategy, Stock futures pricing, Empirical study

1 Introduction

It is significant for a modern enterprise to have an effective system of risk management and financial innovation. One of the approaches is taking advantage of various attainable derivatives, which mainly including futures in China. Therefore, it is necessary to be acquainted with that the futures are accurately pricing.

In complete markets, there exists unique price for every capital asset. However, the actual financial market is incomplete[1], which means there are different prices determined through risk-neutral pricing method under arbitrage-free conditions. In technical terms, the problem is that no unique martingale measure exists[2]. A variety of principles to select one specific martingale measure have been suggested for the purpose of pricing uniquely.

One of the principles is originated from Föllmer and Sondermann (1986). They first introduced the concept of the “risk-minimizing strategy”[3] which enables the investor to minimize the risk exposure of the portfolio at any time before the terminal time. Föllmer and Schweizer (1991) proposed the notion of the “minimal martingale measure” and associated it to the risk-minimizing strategy[4].

Most researches have tended to focus on continuous-time incomplete markets, rather than on discrete-time incomplete markets. He (1998) initiated the research in discrete-time incomplete markets with the introduction of the locally risk-minimizing strategy[5]. Li (2000) characterized the minimal martingale measure in a specified discrete-time incomplete market model[6]. Nevertheless, although several researches have been devoted to contingent claim pricing in discrete-time markets, rather less attention has been paid to pricing actual financial derivatives. This makes the research less applicable to actual financial markets.

Considering every order occurs discretely, actual financial markets are deemed to be in discrete time and also incomplete as stated before. Moreover, for realistic application, it is essential to extend the research to actual derivatives, such as futures, the main kind of derivatives in China. Therefore, the present investigation attempts to develop the research of futures pricing in discrete-time incomplete markets. The remainder of this paper is divided into three sections: In the next section, we recall the definition of several basic concepts and work out the minimal martingale measure in another specified market model.

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Under the measure we derive the arbitrage-free pricing model of stock futures used in this paper. In section 3, we state and discuss our main results on the basis of an empirical study. Section 4 contains the brief conclusions.

2 Models

2.1 Basic concepts: the locally risk-minimizing strategy and the minimal martingale measure

Consider a discrete-time incomplete market, where there are $d$ kinds of risky assets and one risk-free asset. Let $(\Omega, F, (F_n)_{0 \leq n \leq N}, P)$ be a filtered probability space, where we assume that the filtration $(F_n)_{0 \leq n \leq N}$ satisfies the usual assumptions. We denote with $(S_n^0)_{0 \leq n \leq N}$ (let $S_0^0 = 1$) the price process of the risk-free asset and with $(S_n^\phi)_{0 \leq n \leq N} = ((S_n^\phi), \cdots, S_n^\phi)$ the price process of the $d$ kinds of risky assets. Let $\beta_n = (S_n^0)^{-1}$ be the discount factor, then the discounted price process of the risk-free assets should be $(\tilde{S}_n)_{0 \leq n \leq N} = (\beta_n S_n^0)_{0 \leq n \leq N}$. Suppose $E[|\tilde{S}_n^i|^2] < \infty$ for $0 \leq n \leq N$, then $(\tilde{S}_n)$ has the Doob decomposition as $\tilde{S}_n = \tilde{S}_0 + \tilde{W}_n + \tilde{A}_n$, $0 \leq n \leq N$, where $(\tilde{W}_n)_{0 \leq n \leq N}$ (resp. $(\tilde{A}_n)_{0 \leq n \leq N}$) is a $d$-dimensional square-integrable martingale (resp. predictable process) with initial value set as zero.

Because there exist unattainable contingent claims which cannot be replicated by any normal self-financing strategy, the concept of the traditional trading strategy should be extended. An extended trading strategy is a $R^{d+1}$-valued stochastic process $(\phi_n)_{0 \leq n \leq N}$ where $\phi_n = (\phi_0^n, \phi_1^n, \cdots, \phi_d^n)$, $\phi_0^n$ (resp. $\phi_i^n$) represents the number of units of the risk-free asset (resp. the $i$th risky asset) held at time $n$. Let $(V_\psi(\phi))_{0 \leq n \leq N}$ and $(\tilde{V}_\psi(\phi))_{0 \leq n \leq N}$ denote the wealth process and the discounted wealth process of the strategy respectively. We define that the set $\Psi$ consists of all extended trading strategies and the set $L^1_c(F)(L_2$ for short) consists of all $F$-measurable 1-dimensional square-integrable stochastic variables.

For the contingent claim $X \in F_N^+$, $\tilde{X} = \beta X \in L_2$, if an extended trading strategy $\phi \in \Psi$ satisfies $V_\psi(\phi) = \tilde{X}$, it is called the extended hedging strategy of $X$. He (1998) stated that there always exists an extended trading strategy of a contingent claim$^4$.

Moreover, we have stated the self-financing strategy is not sufficiently reasonable in our case. Therefore, for the more accurate assumption of mean-self-financing strategy, we suppose $\phi = (\phi^0, \phi) \in \Psi$ and let

$$C_n(\phi) \doteq \tilde{V}_n(\phi) - \sum_{i=1}^n \phi_i \Delta \tilde{S}_i, \quad 0 \leq n \leq N; \quad R_n(\phi) \doteq E \left[ (C_n(\phi) - C_n(\phi^0))^2 | F_n \right], \quad 0 \leq n \leq N,$$

where $(C_n(\phi))_{0 \leq n \leq N}$ is called the discounted cost process, and $R_n(\phi)$ is used to measuring the risk level of the trading strategy. If $(C_n(\phi))_{0 \leq n \leq N}$ is a square-integrable martingale, $\phi$ is called the mean-self-financing strategy. Use $\Psi_m$ for the set of the mean self-financing strategies.

**Definition 2.1** For any $\phi = (\phi^0, \phi) \in \Psi$, $j \in \{0, 1, \cdots, N-1\}$, if $V_\psi(\phi) = V_n(\phi)$ and $\phi_n = \phi_j$, $n \notin \{j, j + 1\}$ is the sufficient condition of $R_j(\phi) \leq R_j(\phi), \quad j = 0, 1, \cdots, N$, then $\phi = (\hat{\phi}, \hat{\phi}) \in \Psi$ is called the locally risk-minimizing trading strategy.

It has been proved that for contingent claim $X \in F_N^+, \tilde{X} \in L_2$, a locally risk-minimizing strategy $\hat{\phi} \in \Psi_m$ is an extended hedging strategy if $\tilde{X}$ has Föllmer-Schweizer decomposition$^{[4,6]}$.

**Definition 2.2** A minimal martingale measure is an equivalent martingale measure satisfying
1) Any square-integrable martingale orthogonal to \((\tilde{W}_n)\) is a \(Q\)-martingale; 2) \(E\left[ \frac{dQ}{dP} \right] < \infty\),

where we denote with \(P\) the objective measure or historical measure, which is always not specially noted after any expectation symbol \(E\), while the subjective measure or risk-neutral measure \(Q\) is noted for distinction.

It follows from Li(2000) that if there exists a minimal martingale measure, it will be unique. And the following lemma gives the relationship between the minimal martingale measure and the locally risk-minimizing hedging strategy\(^{[6]}\).

**Lemma 2.1** Suppose there exists a minimal martingale measure \(Q \in \mathbb{P}\), if \(\phi = (\hat{\phi}_n, \hat{\phi}) \in \Psi_n\) is the locally risk-minimizing extended hedging strategy of contingent claim \(X \in F_n\), \(\tilde{X} = \beta X \in L^1\), we will have \(\tilde{V}_n(\hat{\phi}) = E_Q[X\beta_n|F_n]\), \(0 \leq n \leq N\). Here \(\mathbb{P}\) is the set of all equivalent martingale measures in incomplete markets.

### 2.2 Characterization of the minimal martingale measure in a specified market model

In the following we give an explicit representation of the minimal martingale measure in a specified discrete-time incomplete market model. The simplifying assumption should be added to the market model described in section 2, because there is certain difficulty in mathematical solution under the situation of \(d\) kinds of risky assets. So in the following sections, we regard the market consists of only two assets: the portfolio asset which is deemed to have contained the messages of all the risky assets in the market approximately and the risk-free asset. Let \((M_n)_{0 \leq n \leq N}\) and \((R_n)_{1 \leq n \leq N}\) denote the price process and the return process of the portfolio asset, while \((r_n)_{1 \leq n \leq N}\) is the return process of the risk-free asset. Furthermore, following the said assumption, we reckon the minimal martingale measure derived in this simplified model to be the proxy of the true one in actual market with \(d\) kinds of risky assets.

From the properties of the minimal martingale measure\(^{[6]}\) and Doob decomposition of \((M_n)_{0 \leq n \leq N}\), we can derive the following theorem.

**Theorem 2.1** The probability measure \(Q\) defined by \(dQ = Z_N dP\) is a minimal martingale measure if and only if \(\alpha_1\) and \(\alpha_n\) are set as

\[
\alpha_1 = \frac{(1 + r_1) E[M_1]}{M_0 Var(R_1)}, \quad \alpha_n = \frac{(1 + r_n)(r_n - E[R_n])}{M_{n-1} Var(R_n)}, \quad 2 \leq n \leq N, \tag{2.1}
\]

and \((Z_n)\) satisfies

\[
Z_n = \sum_{k=1}^{n} (1 + \alpha_k \Delta M_k)
= \left(1 + \frac{(1 + r_1)(1 + E[R_1])(R_1 - E[R_1])}{Var(R_1)} \right) \times \prod_{k=2}^{n} \left(1 + \frac{(r_k - E[R_k])(R_k - E[R_k])}{Var(R_k)} \right) \quad 1 \leq n \leq N. \tag{2.2}
\]

### 2.3 Stock futures pricing models

In this subsection, the contingent claim pricing is extended to the stock futures pricing. Before the introduction of the futures, it is very significant to add the following assumptions: (1) As a kind of risky asset, the introduction of one futures into the market described in subsection 2.2 will not have an effect on the market’s basic properties, especially on the incompleteness of the market. (2) The minimal martingale measure determined in subsection 2.2 will not be changed after the introduction of one kind of futures. The said assumptions are reasonable through the scope comparison between one asset and the whole market. Now suppose there is a stock futures contract expiring at time \(N\), and the investor wants to know the value of this contract at time \(n\). In another words, \(F_{n,N}, 0 \leq n < N\), the price at time \(n\), is
our target. Let \((\hat{S}_n)_{n\in\mathbb{N}}\) and \((\hat{R}_n)_{n\in\mathbb{N}}\) be the price process and the return process of the underlying stock respectively. From the arbitrage-free futures pricing formula under the martingale measure\(^7\), the intrinsic logic of which is consistent with lemma 2.1, we have

\[
F_{n,N} = E_{\hat{F}}(F_{N,N} | F_n) \Rightarrow F_{n,N} = \frac{1}{Z_n} E\left[F_{N,N}Z_N | F_n\right]
\]  

(2.3)

Here, the deduction follows the equation \(E_{\hat{F}}(X | F) = \frac{1}{E(Z_N | F)} E(Z_N X | F) \) and \(E(Z_N | F) = Z_n\).\(^8\)

Moreover, because the futures price converges at maturity date, we obtain \(F_{N,N} = \hat{S}_N\). Substituting it and \((Z_n)\) from (2.2) into (2.3), we have

\[
F_{n,N} = \frac{1}{Z_n} E[\hat{S}_N Z_N | F_n]
\]

\[
= E[\hat{S}_n(1 + \hat{R}_{n+1}) \cdots (1 + \hat{R}_N) \prod_{k=n+1}^{N} \left(1 + \frac{(r_k - E[R_k])(R_k - E[R_k])}{\text{Var}(R_k)}\right)] | F_n]
\]

(2.4)

\[
= \hat{S}_n E[n+1 \prod_{k=n+1}^{N} \left(1 + \frac{(r_k - E[R_k])(R_k - E[R_k])}{\text{Var}(R_k)}\right)(1 + \hat{R}_k)]
\]

For simplicity, we suppose \(r_n\) is a constant denoted by \(r\), and let the price processes of the underlying stock and the portfolio asset both follow the Geometric Brown Motion. For general reference we refer to Hull (2001, 4th edition)\(^9\). Therefore, we have

\[
\frac{\Delta M}{M_n} = \mu N + \delta \sqrt{N}, \quad \frac{\Delta \hat{R}}{\hat{S}_n} = \hat{\mu} N + \hat{\delta} \sqrt{N}, \quad 0 \leq n \leq N
\]

(2.5)

where \(\varepsilon\) follows the standard normal distribution; \(\mu\) (resp. \(\hat{\mu}\)) and \(\delta\) (resp. \(\hat{\delta}\)) denote the expectation and the standard deviation of the return of the portfolio asset (resp. the underlying asset) respectively. Thus (2.5) can be simplified as

\[
F_{n,N} = \hat{S}_n \left\{E \prod_{k=n+1}^{N} \left[1 + \frac{(r - \mu)(R_k - \mu)}{\delta^2}\right](1 + \hat{R}_k)\right\}
\]

\[
= \hat{S}_n \left[1 + \hat{\mu} + \frac{(r - \mu) \text{cov}(R_k, \hat{R}_k)}{\delta^2}\right]^{N-n}, \quad 0 \leq n \leq N.
\]

(2.6)

The above deduction makes use of the I.D.D. assumption of each return series. Formula (2.6) shows the general form of our stock futures pricing models in the specified market model. In the next section, we test the efficiency of the models using the data of several representative stock futures and stock index futures.

3 Empirical Study

3.1 Sample futures pricing models

This study uses the futures price data on the Dow Jones Index (DJI for short), one typical banking stock Citi Group (Citi for short) and one typical consumer goods stock Coca-cola (KO for short). We choose the S&P 500 stock index as the proxy for the portfolio asset, because its 500 component stocks, chosen from various industries, are quite representative for the whole market. The US 3-month Treasury bill secondary market rate is used as the proxy for the risk-free interest rate. All data are obtained from Bloomberg database and http://cn.finance.yahoo.com/, covering the period 1 May 2005 to 1 June 2006.
Using the above data, we obtain the annual return series of each sample. Table 1 reports the summary statistics for the series. Substituting them into (2.6), we have the specified form of pricing models for each sample futures. Take DJI index futures for example. Its pricing model can be determined as

\[
F^\text{DJI}_{n,N} = \hat{S}^\text{DJI}_n \left[1 + \mu + \frac{(r - \mu) \text{cov}(R_n, \hat{R}^\text{DJI}_n)}{\delta^2} \right]^{N-n} = \hat{S}^\text{DJI}_n \left[1 + 0.0615 + \frac{(0.0394 - 0.0610) \times 0.5206}{0.4928^2} \right]^{N-n}
\]

(3.1)

Similarly, the futures pricing models of KO and Citi can be determined respectively as

\[
F^\text{KO}_{n,N} = \hat{S}^\text{KO}_n (1 + 0.02623)^{N-n}, \quad F^\text{Citi}_{n,N} = \hat{S}^\text{Citi}_n (1 + 0.0329)^{N-n}
\]

(3.2)

### Table 1 Summary statistics of annual returns

<table>
<thead>
<tr>
<th></th>
<th>r</th>
<th>S&amp;P500</th>
<th>DJI</th>
<th>KO</th>
<th>Citi</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Expectation of Annual Return</strong></td>
<td>0.0394</td>
<td>0.0610</td>
<td>0.0615</td>
<td>0.0537</td>
<td>0.0706</td>
</tr>
<tr>
<td><strong>Standard Deviation of Annual Return</strong></td>
<td>0</td>
<td>0.4928</td>
<td>0.5519</td>
<td>0.7232</td>
<td>1.1396</td>
</tr>
<tr>
<td><strong>Covariance with S&amp;P500</strong></td>
<td>0</td>
<td>0.4928</td>
<td>0.5206</td>
<td>0.6235</td>
<td>0.8567</td>
</tr>
</tbody>
</table>

### 3.2 Efficiency test

According to the pricing models determined in subsection 3.1, we calculate the predicted price series of each sample futures. In order to test the efficiency of the pricing models, a comparison between predicted prices and actual prices (only those closing prices that are active in market and near to maturity are considered) is carried out. The results of the comparison, shown in Figure 1 to Figure 3, reveal that the prices predicted by these pricing models could fit the actual prices well, and fluctuations of two price series are almost identical in trend.

To further investigate the efficiency of new pricing models, we add the price series predicted by another traditional futures pricing model, single-factor model of cost-of-carry theory for complete market, into Figure 1 to Figure 3. The comparison of two kinds of the predicted price series is also displayed in these figures. We can see that the two series of each sample are in the same trend and cross with each other, which make it difficult to judge perceptively which model is better. For clearer judgment, we calculate the relative deviations of two models between predicted prices and actual prices of each sample and classify them into several small intervals. The results are shown in Table 2 and Table 3.

![Figure 1 Comparison of prediction efficiency (DJI)](image-url)
From the results in Table 2 and Table 3, we get two indications and consequent analyses:

1) For the stock index futures, the new model and the traditional single-factor model are almost of same efficiency in price prediction. The statistical results of the prediction deviations of two models are almost identical, not only from the comparison of the distributions of the relative deviation, but also from the comparison of the sum squared relative deviations. To illustrate this result, we trace back to the
settings of our model. We can observe that the portfolio asset set in our model is highly correlative with the index, and the index futures could be hedged almost efficiently by the index. Therefore, the index futures might be almost attainable by the portfolio asset, which makes the index futures pricing nearly identical with the case in complete markets. In technical terms, we have $\hat{\mu}_\text{DJI} \approx \mu$ and $\text{cov}(R_n, \hat{R}_n^\text{DJI}) \approx \delta^2$, thus the pricing model of DJI index futures could be approximately simplified as

$$F_{n,N}^{\text{DJI}} = \delta_n \left[ 1 + \hat{\mu}_\text{DJI} + \frac{(r - \mu) \text{cov}(R_n, \hat{R}_n^\text{DJI})}{\delta^2} \right]^{N-n} \approx \delta_n \left[ 1 + \hat{\mu}_\text{DJI} + r - \mu \right]^{N-n}$$

(3.3)

From (3.3), it is easy to see that the approximately simplified model is as the same as the traditional single-factor model. This also supports the explanations of the test result.

2) For the stock futures, there is a strong possibility that the new model performs better than the traditional single-factor model in prediction. As can be seen in Table 2, in the interval of the smallest relative deviation “less than 0.005”, the number of the prices predicted by the new model is obviously larger than the one of the single-factor model; while in the largest deviation interval the situation is just opposite. Therefore, it is reasonable to accept that the new model is very likely to be more reliable. Further, Table 3 shows that for all the samples except DJI, the Sum Squared Relative Deviations (SSRDs for short) of predictions by the new model, both less than 0.0025, are less than the SSRDs of predictions by the single-factor model, both larger than 0.005. It indicates a relatively better prediction precision of the new model than the one of the traditional single-factor model. The results could be explained from the view of financial economics. Considering the unattainable assets, the new model is derived under the incomplete market assumption, which is close to the actual situation of financial markets just as stated in the introduction of this paper; while the traditional single-factor model is based on the complete market assumption. Therefore, that the prediction by the new model is better than the one of the traditional single-factor model both in reliability and precision is highly acceptable.

4 Conclusion

This paper derives the stock futures pricing model based on the minimal martingale measure in a specified discrete-time incomplete market model. Using the futures price data on the Dow Jones Index, stock Citi Group and stock Coca-cola, we investigate the efficiency of the model. The results lead to the following conclusions:

1) For the samples of stock futures and stock index futures, the prices predicted by the new model could fit the actual prices well;
2) Compared to the traditional single-factor model of cost-of-carry theory for complete markets, it is very probably that the new model predicts the prices of stock index futures of almost equal efficiency, and performs better for the prediction of stock futures prices in both reliability and precision. Therefore, it is reasonable to accept that the new model could be a useful approach for pricing stock futures and stock index futures. To develop the model’s applicability, further improvement of the model, such as introducing “dividend process” to price special financial futures and considering “convenience yield” or other factors into pricing commodity futures, is an area for future researches.

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